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Tensor Products of Quantum Structures and Their Applications in Quantum Measurements

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Tensor products of quantum logics and effect algebras with some known problems are reviewed. It is noticed that although tensor products of effect algebras having at least one state exist, in the category of complex Hilbert space effect algebras similar problems as with tensor products of projection lattices occur. Nevertheless, if one of the two coupled physical systems is classical, tensor product exists and can be considered as a Boolean power. Some applications of tensor products (in the form of Boolean powers) to quantum measurements are reviewed.

KEY WORDS: quantum logic; orthomodular lattice; effect algebra; Hilbert space effects; tensor product; quantum measurement.

1. TENSOR PRODUCTS OF QUANTUM LOGICS

In the quantum logic approach to quantum mechanics, the set of all events is modeled by an abstract algebraic structure called a *quantum logic* (Birkhoff and von Neumann, 1936). Most usually, the quantum logic is supposed to be a σ-orthomodular poset, resp. lattice (Varadarajan, 1985). Recall that an orthomodular poset (OMP) L is a bounded partially ordered set with the smallest elements 0 and the greatest element 1, endowed with an orthocomplementation $' : P \to P$ such that (i) $a \leq b \Rightarrow b' \leq a'$, (ii) $a'' = a$, $a \vee a' = 1$, which satisfies the orthomodular law $a \leq b \Rightarrow b = a \vee (a' \wedge b)$ (it is supposed that all involved lattice operations exist). If, in addition, $\vee_i a_i$ exists in L for any sequence $(a_i)_i$ of elements such that $a_i \leq a_j'$ whenever $i \neq j$, we obtain *σ-orthomodular poset* (σ-OMP). An OMP, which is a lattice, is an orthomodular lattice (OML).

We call elements $a, b \in L$ *orthogonal*, written $a \perp b$, if $a \leq b'$. Let P and Q be (σ-) orthomodular posets. A mapping ϕ: $P \to Q$ is a (σ-) *morphism* if $\phi(1) = 1$, $\forall a \in P$, $\phi(a') = \phi(a)'$, and for every finite (countable) pairwise orthogonal family $(a_i)_i$, $\phi(\vee_i a_i) = \vee_i \phi(a_i)$. A bijective morphism is an *isomorphism* if the inverse

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mapping ϕ^{-1} is also a morphism. A morphism preserving all existing joins and meets will be called a *homomorphism*.

A (σ-additive) *state* on L is a mapping $s: L \to \mathbb{R}^+([0,1])$ such that (i) $s(1) = 1$ and (ii) $\forall a, b \in L, a \perp b \Rightarrow s(a \vee b) = s(a) + s(b)$ (for every pairwise orthogonal sequence $(a_i)_i$, such that $\vee_i a_i \in L$, $s(\vee_i a_i) = \sum_i s(a_i)$). A set of states M on L is called *ordering* if $m(a) \leq m(b)$ for all $m \in M$ implies $a \leq b$ $(a, b \in L)$.

Let S_1 and S_2 be two physical systems described by quantum logics P and Q, respectively. To describe a composite system $S_1 + S_2$, we need a quantum logic, which we denote by $P \otimes Q$, with some desirable properties. Such properties were formulated in Foulis and Randall (1981) as follows:

(i) $P \otimes Q$ is an OMP.
(ii) $\otimes$ is a map from $P \times Q$ to $P \otimes Q$ such that, $p_1 \otimes q_1 \perp p_2 \otimes q_2$ if either $p_1 \perp p_2$ or $q_1 \perp q_2$.
(iii) If α and β are states on P and Q, respectively, then there exists a state γ on $P \otimes Q$ such that $\gamma(P \otimes Q) = \alpha(p)\beta(q)$ for all $p \in P$ and all $q \in Q$.

The following counterexample found by Foulis and Randall (1981) shows that such $P \otimes Q$ may not exist. Let us consider the "pentagon," i.e., an OMP L with the Greechie diagram (Pták and Pulmannová, 1991) consisting of the blocks

$$\{a, x, b\}, \{b, y, c\}, \{c, z, d\}, \{d, u, e\}, \{e, v, a\}.$$

L is an OML in fact, and has an ordering set of states. Put $P = Q = L$, and assume that $P \otimes Q$ with the properties listed above exists. Then the set $D = \{a \otimes a, b \otimes c, c \otimes e, d \otimes b, e \otimes d\}$ consists of pairwise orthogonal elements. Consider the state α on P such that $\alpha(a) = \alpha(b) = \alpha(c) = \alpha(d) = \alpha(e) = \frac{1}{2}$, and $\alpha(x) = \alpha(y) = \alpha(z) = \alpha(u) = \alpha(v) = 0$. Put $\beta = \alpha$, then in the state γ we have $\gamma(\vee D) = \frac{5}{4} > 1$, a contradiction.

In the Hilbert space approach to quantum mechanics, the quantum logic corresponds to the set of all closed linear subspaces of a Hilbert space H, or equivalently, to the set of all orthogonal projections on H, which is called a Hilbert lattice denoted by $P(H)$. Tensor products in the category of Hilbert lattices were studied by Malolcsi, (1975) and Aerts and Daubechies (1978). The definition is as follows (Malolcsi, 1975):

Let H_1 and H_2 be Hilbert spaces, both complex or both real. For a Hilbert space H, $(P(H); u_1, u_2)$ is called a *tensor product* of $P(H_1)$ and $P(H_2)$ if

(i) $u_1: P(H_i) \to P(H)$ is a σ-homomorphism $(i = 1, 2)$,
(ii)

$$\bigvee_{n=1}^{\infty} \bigvee_{m=1}^{\infty} \left(u_1\left(M_1^n\right) \wedge u_2\left(M_2^m\right)\right) = \left(\bigvee_{n=1}^{\infty} u_1\left(M_1^n\right)\right) \wedge \left(\bigvee_{m=1}^{\infty} u_2\left(M_2^m\right)\right)$$

for any pairwise orthogonal elements M_1^n of $P(H_1)$ and any pairwise orthogonal elements M_2^m of $P(H_2)$,

(iii) $u_1(P(H_1))$ and $u_2(P(H_2))$ generate $P(H)$, that is the smallest σ-OML of subspaces containing both $u_1(P(H_1))$ and $u_2(P(H_2))$ is $P(H)$.

Let $(P(H));\ u_1, u_2)$ and $(P(H'); u_1', u_2')$ be tensor products of $P(H_1)$ and $P(H_2)$. We say that $(P(H'); u_1', u_2')$ is *subordinated* to $(P(H);\ u_1, u_2)$ if there is a σ-homomorphism $u : P(H) \to P(H')$ such that $u_i' = u \circ u_i (i = 1, 2)$. If $(P(H);\ u_1,\ u_2)$ is also subordinated to $(P(H); u_1', u_2')$ then the two tensor products are said to be *equivalent*. It was proved that the only possible subordination between tensor products of Hilbert space lattices is equivalence (Malolcsi, 1975).

Let $M_2 \in P(H_2)$, $M_2 \neq 0$ be fixed. The map $f_{1,M_2} : P(H_1) \to P(u_2(M_2))$ defined by

$$f_{1,M_2} = u_1(M_1) \wedge u_2(M_2)\,(M_1 \in P(H_1))$$

is a σ-homomorphism. The same is true for the map f_{2,M_1} defined similarly for a fixed nonzero element M_1 of $P(H_1)$.

If we add the following *condition of fullness*: the σ-homomorphisms $f_{1,[x_2]}$ and $f_{[x_1],1}$ are surjective for all nonzero $x_2 \in H_2$ and $x_1 \in H_1$ (where $[x]$ denotes the one-dimensional subspace corresponding to a vector x), we obtain the following result.

Theorem 1.1. *Let H_1 and H_2 be Hilbert spaces, dim $H_1 \geq 3$, dim $H_2 \geq 3$. If the Hilbert spaces are complex, then there exist exactly two (nonequivalent) tensor products of $P(H_1)$ and $P(H_2)$ satisfying the condition of fullness. They are given by*

(i) $H = H_1 \otimes H_2, u_1(M_1) = M_1 \otimes H_2, u_2(M_2) = H_1 \otimes M_2$;

(ii) $H = \overline{H}_1 \otimes H_2, u_1(M_1) = \overline{M}_1 \otimes H_2, u_2(M_2) = \overline{H}_1 \otimes M_2$; where $\overline{K}$ denotes the conjugate Hilbert space of a Hilbert space K, and $\otimes$ denotes the usual tensor product of Hilbert spaces.

If the Hilbert spaces are real, there is only one tensor product of $P(H_1)$ and $P(H_2)$ satisfying the condition of fullness. It can be obtained from the above formulae, taking the case (i).

A similar result was obtained in Aerts and Daubechies (1978), where the problem was studied in a more general context.

The problems with tensor products were one of the reasons to replace orthomodular posets by more general structures.

2. TENSOR PRODUCTS OF EFFECT ALGEBRAS

Effect algebras were introduced as an abstraction of the Hilbert space effects, i.e., self-adjoint operators between the zero operator 0 and the identity operator I on a Hilbert space H. These operators play an important role in the theory of quantum measurements, because quantum mechanical observables, represented by positive operator valued measures, have their ranges in the set $\mathcal{E}(H)$ of the Hilbert space effects.

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations 0, 1 satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.
(E4) If $a \oplus 1$ exists, then $a = 0$.

In an effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It is easy to check that $\leq$ is a partial order on E. In this partial order, 0 is the least and 1 is the greatest element of E. Moreover, it is possible to introduce a new partial operation $\ominus$; $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$. Therefore, it is usual to denote the domain of $\oplus$ by $\perp$. We say that elements a and b in an effect algebra E are *orthogonal* if $a \perp b$. In what follows, when we write $a \oplus b$ we mean that $a \oplus b$ is defined (i.e., $a \perp b$). Owing to associativity (E2), we may omit parentheses in $a_1 \oplus a_2 \oplus a_3$ and $a_1 \oplus a_2 \oplus \cdots \oplus a_n$, the latter term being defined by induction. We will say that the elements $a_1, \ldots, a_n$ are **orthogonal** if the element $a_1 \oplus \cdots \oplus a_n$ exists in L. More generally, we say that $\{a_\alpha\}_\alpha$ is an *orthogonal family* if every finite subfamily is orthogonal.

An effect algebra L is called (σ)-*orthocomplete* if for every (countable) indexed family $\mathcal{A} := (a_\alpha)_\alpha$ of elements of L such that its every finite subfamily is orthogonal, the element $\oplus\mathcal{A} := \bigvee_F \oplus\{a : a \in F\}$ is defined, where the supremum goes over all finite subfamilies of $\mathcal{A}$.

It turns out that the class of effect algebras contains all known structures used so far as quantum logics—orthomodular lattices, orthomodular posets, orthoalgebras, Boolean algebras—as special subclasses, and it also contains MV-algebras, which were introduced as an algebraic basis of many-valued logic.

The following relations between effect algebras and abelian groups were proved in (Foulis and Bennett, 1994).

Let G be a partially ordered abelian group, let $0 \neq u \in G^+$, and let $L = G^+[0, u] := \{g \in G^+ : 0 \leq g \leq u\}$. Then L can be organized into an effect algebra by defining $p \oplus q$ iff $p + q \in L$, in which case $p \oplus q = p + q$. In the effect algebra L, we have $p' = u - p$ and the effect partial order on L coincides with the restriction to L of the partial order on G. An effect algebra of the form $G^+[0, u]$ as described above is called an *interval effect algebra*. Important examples of interval effect algebras are $\mathbb{R}^+[0, 1]$, the unit interval of the real line, and the set of Hilbert space effects, i.e., self-adjoint operators in the interval $[0, I]$ on a Hilbert space.

Let E be an effect algebra, G an Abelian group. A map $\rho : E \to G$ is a G-valued measure on E if $p \perp q \Rightarrow \rho(p \oplus q) = \rho(p) + \rho(q)$ for all p, $q \in E$.

Let L be an effect algebra. By a *universal group* for L, we mean a pair (G, γ) consisting of an Abelian group G and a G-valued measure $\gamma : L \to G$ such that the following conditions hold.

(i) $\gamma(L)$ generates G.
(ii) If K is an Abelian group and $\phi : L \to K$ is a K-valued measure, then there is a group homomorphism $\phi_* : G \to K$ such that $\phi = \phi_* \circ \gamma$.

If a universal group exists, it is unique up to isomorphism.

Proposition 2.1. ***[9, Theorem 9.2]*** *If L is an effect algebra, then there is a universal group (G, γ) for L.*

Moreover, L is an interval effect algebra iff it has a partially ordered universal group (G, γ) with G^+ consisting of all finite $\oplus$-sums of elements of $\gamma(L)$, and L is isomorphic with the interval $G^+[0, \gamma(1)]$ in G. For more details on effect algebras see Dvurečenskij and Pulmannová (2000) and citations therein.

Let us consider structures consisting of sets endowed with a (everywhere defined) commutative, associative operation $+$ with zero element 0 (commutative monoids, Abelian groups), and possibly with a partial ordering $\leq$ (partially ordered Abelian groups).

If A, B, C are structures, and $f : A \times B \to C$, we say that f is a *bimorphism* (Wehrung, 1996) when for all $a \in A$ (resp. $b \in B$) the map $f(a, .)$ (resp. $f(., b)$) is a homomorphism of monoids. If $\leq$ is defined in A, B, C, we say that f is *positive* when for all positive $a \in A$ and $b \in B$, we have $f(a, b) \geq 0$. We say that the [positive] bimorphism f is *universal* (relative to a given category of structures) when for every structure D and every [positive] bimorphism $g : A \times B \to D$, there exists a unique [positive] homomorphism $\overline{g} : C \to D$ such that $\overline{g} \circ f = g$, in this case the pair (C, f) is unique up to isomorphism and the custom is to call it the *tensor product* of A and B, written $C = A \otimes B$, $f(a, b) = a \otimes b$. This notion is very sensitive to the category of structures under consideration. Notice that for all the three categories above, the tensor product exists.

The tensor product of Abelian groups does not preserve all inner structure of the given groups. In Wehrung, 1996, Example 1.5], two torsion-free directed interpolation groups A and B are constructed such that $A \otimes^{\mathrm{oag}} B$ is not an interpolation group. Even more surprising result is [Wehrung, 1996, Example 1.6]: $\mathbb{R} \otimes^{\text{oag}} \mathbb{R}$ in the category of partially ordered Abelian groups is not lattice-ordered.

Now we will consider tensor products of effect algebras. We will need the following definitions.

Let E, F be effect algebras. A mapping $\phi : E \to F$ is said to be

(i) *additive* if $p, q \in E,\ p \perp q \Rightarrow \phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$,
(ii) a *morphism* if it is additive and $\phi(1) = 1$,
(iii) a *positive* morphism if ϕ is a morphism and $p \in E$ with $\phi(p) = 0$ implies $p = 0$,
(iv) a *homomorphism* if ϕ is a morphism and $p, q \in E, \exists p \wedge q \Rightarrow \phi(p) \wedge \phi(q) = \phi(p \wedge q)$,
(v) a *monomorphism* if ϕ is a morphism and $p, q \in E, \phi(p) \leq \phi(q) \Rightarrow p \leq q$,
(vi) an *isomorphism* if ϕ is a surjective monomorphism.

A *state* on E is a morphism $\mu : E \to \mathbb{R}^+[0, 1]$ from E into the unit interval of $\mathbb{R}$. A set $\mathcal{P}$ of states on L is *ordering* if for $a, b \in L$, $m(a) \leq m(b)$ for all $m \in \mathcal{P}$ implies $a \leq b$.

Let P, Q, and L be effect algebras. A mapping $\beta : P \times Q \to L$ is called a *bimorphism* if, for each $p \in P$ and each $q \in Q$, $\beta(p, .)\ Q \to L$ and $\beta(., q) : P \to L$ are additive and $\beta(1, 1) = 1$.

If E and F are effect algebras and $\phi : E \to F$ is a morphism, then $\phi(0) = 0$ and for $p, q \in E,\ p \leq q \Rightarrow \phi(p) \leq \phi(q)$ with $\phi(q \ominus p) = \phi(q) \ominus \phi(p)$. In particular, $\phi(p') = \phi(p)'$. A morphism $\phi : E \to F$ is a monomorphism iff $\phi(p) \perp \phi(q) \Rightarrow p \perp q$ for all $p, q \in E$. Also, if $\phi : E \to F$ is an isomorphism, then ϕ is a bijection and ϕ^{-1} is an isomorphism. A morphism is called a σ-morphism if it preserves all existing countable $\oplus$-sums.

The following definition of tensor products of effect algebras is analogous to the tensor products of Abelian groups. This analogy was first used in Foulis and Bennett, 1993 for the construction of tensor products of orthoalgebras, and it was extended in Dvurečenskij (1995) to tensor products of effect algebras.

Let P, Q, and L be effect algebras. Let there be a bimorphism $\theta : P \times Q \to L$ such that the following conditions are satisfied:

(T1) If $\beta : P \times Q \to K$ is a bimorphism, where K is any effect algebra, then there exists a morphism $\phi : L \to K$ such that $\beta = \phi \circ \theta$.
(T2) Elements of the form $\theta(p, q)$, $p \in P, q \in Q$ generate L (i.e., every element of L is a finite $\oplus$-sum of elements of the form $\theta(p, q)$).

The effect algebra L described above is called the tensor product of the effect algebras P and Q, and we usually write $L = P \otimes Q$, $\theta(p, q) = p \otimes q$.

In Dvurečenskij (1995), the following result was obtained.

Proposition 2.2. *Let P, Q be effect algebras. If there is an effect algebra A and a bimorphism $\alpha : P \times Q \to A$, then tensor product of P and Q exists.*

In particular, if there are states r and s on P and Q, respectively, then the tensor product $P \otimes Q$ exists. Indeed, if we define $\beta(p, q) = r(p)s(q)$, then $\beta : P \times Q \to \mathbb{R}^+[0, 1]$ is a bimorphism.

Tensor products of two effect algebras can be generalized in a natural way to tensor products of a finite collection of effect algebras (Pulmannová, 1995).

Some properties of tensor products are collected in the following proposition.

Proposition 2.3.

- *(i) Let A, B, and C be effect algebras. If $(A \otimes B) \otimes C$ exists, then $A \otimes (B \otimes C)$ exists and they are isomorphic (Pulmannová, 1995).*
- *(ii) If D is a collection of effect algebras and for any two elements of D tensor product exists in D, then tensor product of any finite subset of D exists in D (Pulmannová, 1995).*
- *(iii) Suppose that the effect algebras A and B have a tensor product $A \otimes B$ of effect algebras. Let (G_A, α), (G_B, β), and $G_{A\otimes B}, \gamma)$ be the the universal groups of A, B, and $A \otimes B$ with the embeddings α, β, and γ, respectively. Then there is a unique group epimorphism $\tau : G_A \otimes G_B \to G_{A\otimes B}$, such that $\tau(\alpha(p) \otimes \beta(q)) = \gamma(p \otimes q)$ for all $p \in A$, $q \in B$* [?].
- *(iv) Let $\mathcal{A}$ be a family of effect algebras such that for every finite subfamily A_i, $i \leq n$ the tensor product $\bigotimes_{i\leq n} A_i$ exists. Then $(\bigotimes_{i\leq n} A_i)_n$ is a directed family and the direct limit exists.*

We note that the direct limit in (iv) may be considered as an infinite tensor product of the effect algebras in $\mathcal{A}$. It was used in the histories approach to quantum mechanics, (Pulmannová, 1995), (Rudolph, 1996).

According to Foulis, Bennett (1994), every interval effect algebra has at least one state. Therefore, the tensor product of interval effect algebras exists. It is not known if the latter tensor product is an interval effect algebra. But in the class of interval effect algebras, the tensor product exists. Indeed, if G_p and G_Q are universal groups for P and Q, respectively, then $\beta : P \times Q \to (G_P \otimes^{\text{oag}} G_Q)^+[0, u_P \otimes u_Q]$ is a bimorphism. Here $\otimes^{\text{oag}}$ denotes the tensor product in the category of ordered Abelian groups (Wehrung, 1996), and u_P and u_Q are the corresponding units in G_P^+ and G_Q^+, respectively. Using similar arguments to those in Dvurečenskij (1995), we can show that the tensor product in the class of interval effect algebras exists.

Tensor products of Hilbert space effects were considered in Dvurečenskij and Pulmannová (1994). It turns out that the tensor product T of effect algebras $\varepsilon(H_1)$

and $\varepsilon(H_2)$ exists, and is a proper subeffect algebra of $\mathcal{E}(H_1 \otimes H_2)$. Elements of the tensor product consist of finite $\oplus$-sums of elements of the form $A_i \otimes B_i$, where A_i are effects on H_1 and B_i are effects on H_2. So for example, one-dimensional projections corresponding to pure entangled states are not contained in the tensor product. This situation is caused by condition (T2) in the definition of a tensor product, which expresses a purely algebraic way of generation. In addition, the bimorphism τ in the latter definition need not satisfy any special conditions. Therefore, the images $A \to \tau(A, 1)$ and $B \to \tau(1, B)$ of $\varepsilon(H_1)$ and $\varepsilon(H_2)$ into the tensor product T may not reflect properties of these effect algebras in an adequate way.

To improve the situation, the following definitions have been introduced in Gudder (1997).

Definition 2.4. Let P, Q and L be effect algebras. A biomorphism $\beta : P \times Q \to L$ is called left strong if for every nonzero $c \in Q$, $\beta(a, c) \perp \beta(b, c)$ in L if and only if $a \perp b$ in P. Similarly, a bimorphism β is right strong if for every nonzero $a \in P$, $\beta(a, c) \perp \beta(a, d)$ in L if and only if $c \perp d$ in Q. A bimorphism β is called strong if it is left and right strong.

In the category of σ-orthocomplete effect algebras, the notion of a bimorphism is replaced by σ-bimorphism (Gudder, 1998): A mapping $\beta : P \times Q \to L$ is a σ-bimorphism if β is a bimorphism and whenever $a_i \in P$ and $b_i \in Q$ are increasing, then $\beta(\bigvee a_i, b) = \bigvee \beta(a_i, b)$ for every $b \in Q$, and $\beta(a, \bigvee b_i) = \bigvee \beta(a, b_i)$ for every $a \in P$. Equivalently, if $p_i \in P$, $q_i \in q$ are orthogonal sequences, then $\beta(\bigoplus p_i, q) = \bigoplus \beta(p_i, q)$ for all $q \in Q$, and $\beta(p, \bigoplus q_i) = \bigoplus \beta(p, q_i)$ for all $p \in P$. A σ*-tensor product* of P and Q is a pair (T, τ), where T is a σ-orthocomplete effect algebra and $\tau : P \times Q \to T$ is a σ-bimorphism such that the following conditions hold:

T1′ If $\beta : P \times Q \to R$ is a σ-bimorphism into a σ-orthocomplete effect algebra R, then there is a σ-morphism $\phi : T \to R$ such that $\beta = \phi \circ \tau$.

T2′ $\tau(P \times Q)$ generates T, in the sense that the smallest σ-orthocomplete subeffect algebra of T is T.

Definition 2.5. Let P, Q, and L be σ-orthocomlete effect algebras. Effect algebra L will be called a pretensor product of P and Q if the following conditions are satisfied:

(i) There is a strong σ-bimorphism $\tau : P \times Q \to L$ such that $u_1 := \tau(., 1)$ and $u_2 := \tau(1, .)$ are homomorphisms,
(ii) L is generated by $\tau(P \times Q)$.

A pretensor product becomes a tensor product if it has the universal property. That is (L, τ) is the tensor product of P and Q if it satisfies properties (i) and

(ii) of Definition 2.5 and if (K, τ') is another pretensor product, then there is a σ-homomorphism $u : L \to K$ such that $\tau' = u \circ \tau$.

If we add a condition of fullness, i.e., suppose that for every unit vector $r_2 \in H_2$, the mapping $A \to \tau(A, [r_2])$ is surjective, and similarly, for every unit vector $r_1 \in H_1$, $B \to \tau([r_1], B)$ is surjective, we arrive at the following result.

Theorem 2.6. *Let H_1 and H_2 be Hilbert spaces, dim $H_1 \geq 2$, dim $H_2 \geq 2$. Let H be a Hilbert space and $\tau : \varepsilon(H_1) \times \varepsilon(H_2) \to \varepsilon(H)$ be a bimorphism such that $(\varepsilon(H), \tau)$ is a pretensor product. Then*

- *(i) If the Hilbert spaces are complex, then there are exactly two couples (H, τ) which are nonequivalent pretensor products of $\varepsilon(H_1)$ and $\varepsilon(H_2)$ satisfying the condition of fullness. They are given by*
  (1) $H = H_1 \otimes H_2$, $\tau(M_1, M_2) = M_1 \otimes M_2$,
  (2) $H = H_1 \otimes \bar{H}_2$, $\tau(M_1, M_2) = \bar{M}_1 \otimes M_2$,
  where $\otimes$ denotes the usual tensor product of Hilbert spaces and $\bar{K}$ is the dual of the Hilbert space K.
- *(ii) If the Hilbert spaces are real, there is only one pretensor product (hence a tensor product) of $\varepsilon(H_1)$ and $\varepsilon(H_2)$ satisfying the condition of fullness. It can be described as the case (1) above.*

So we obtain a similar result as in the case of the tensor product of Hilbert space quantum logics. For more details see Tensor Products of Hilbert Space Effect Algebras, Preprint.

3. TENSOR PRODUCTS AND QUANTUM MEASUREMENTS

In this paragraph, we reformulate basic features of quantum measurement theory in the frame of quantum logics and effect algebras. Although the reformulation of quantum measurments in the language of quantum logics cannot solve all deep problems of the quantum measurement theory, it may contribute to a better understanding of this problematics.

We first describe elements of a measurement theory in the traditional Hilbert space approach (see Busch *et al.* (1991) for more details). Let a physical system $\mathcal{S}$ be described by a Hilbert space $H_{\mathcal{S}}$, and let X be an observable of $\mathcal{S}$, i.e., a PO-valued measure on H. Further, let $\mathcal{A}$ be a measuring apparatus described by a Hilbert space $H_{\mathcal{A}}$. A *measurment* of X is a 5-tuple $\mathcal{M} = (H_{\mathcal{A}}, X_{\mathcal{A}}, T_{\mathcal{A}}, f, V)$, where $X_{\mathcal{A}}$ is a *pointer observable* (a POV measure on $H_{\mathcal{A}}$), $T_{\mathcal{A}}$ is an initial state of $\mathcal{A}$, f is a *pointer function*, i.e., a measurable function $f : \Omega \mapsto \Omega_{\mathcal{A}}$, which correlates the value spaces $(\Omega, \mathcal{F})$ and $(\Omega_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ of X and $X_{\mathcal{A}}$, respectively, and $V : T(H_{\mathcal{S}} \otimes H_{\mathcal{A}}) \to T(H_{\mathcal{S}} \otimes H_{\mathcal{A}})$ is a trace-preserving positive linear transformation of the trace-class operators $T(H_{\mathcal{S}} \otimes H_{\mathcal{A}})$ of the composite system $\mathcal{S} + \mathcal{A}$, such that the following two requirements are satisfied:

(1) The pointer observable is a classical, that is, an observable which commutes with all other observables on $\mathcal{A}$.
(2) The following equation is satisfied for all $F \in \mathcal{F}$ and for all possible initial states $T \in T(H_{\mathcal{S}_1}^{+})$:

$$tr(TX(F)) = tr(\mathbb{R}_{\mathcal{A}} V(T \otimes T_{\mathcal{A}}) X_{\mathcal{A}} f^{-1}(F)). \tag{1}$$

Here $\mathbb{R}_{\mathcal{A}} V(T \otimes T\mathcal{A})$ denotes the reduction of the final state of $\mathcal{S} + \mathcal{A}$ to $\mathcal{A}$ via relative trace.

If also the equation

$$tr(TX(F)) = tr(\mathbb{R}_{\mathcal{S}} V(T \otimes T_{\mathcal{A}}) X(F))) \tag{2}$$

is satisfied for all $F \in \mathcal{F}$ and all $T \in T(H_{\mathcal{S}})_1^{+}$ the measurement $\mathcal{M}$ is called a first-kind measurement. Here, $\mathbb{R}_{\mathcal{S}} V(T \otimes T_{\mathcal{A}})$ means the reduction of the final state $V(T \otimes T\mathcal{A})$ of $\mathcal{S} + \mathcal{A}$ to the subsystem $\mathcal{S}$.

All features of a measurement $\mathcal{M}$ that pertain to the object system $\mathcal{S}$ are summarized in the instrument $I_{\mathcal{M}}$ of the measurement $\mathcal{M}$. The *instrument* $I_{\mathcal{M}}$ is an operation-valued measure $I_{\mathcal{M}} : \mathcal{F} \to \mathcal{L}(T(H_{\mathcal{S}}))^{+}$, where $\mathcal{L}(T(H_{\mathcal{S}}))^{+}$ is the set of all operations (i.e. positive linear transformation of $T(H_{\mathcal{S}}))^{+}$ defined by

$$I_{\mathcal{M}}(F)T = \mathbb{R}_{\mathcal{S}}(VT \otimes T_{\mathcal{A}}) \cdot I \otimes X_{\mathcal{A}}(f^{-1}(F)), \tag{3}$$

$F \in \mathcal{F}$, $T \in T(H_{\mathcal{S}})$. The instrument reproduces the observable X via the equations

$$tr(TX(F)) = tr(I_{\mathcal{M}}(F)T) \tag{4}$$

for all $F \in \mathcal{F}$, $T \in T(H_{\mathcal{S}})_1^{+}$. An instrument gives the nonnormalized final states $I_{\mathcal{M}}(F)T$ of $\mathcal{S}$ on the condition that the measurement leads to a result in the set F.

Two measurements are called *equivalent* if the corresponding instruments are equal.

A measurement $\mathcal{M}$, or the corresponding instrument $I_{\mathcal{M}}$, is called *repeatable* if for all $E, F \in \mathcal{F}$ and all $T \in T(H_{\mathcal{S}})_1^{+}$ the following equality holds:

$$tr(I_{\mathcal{M}}(E)(I_{\mathcal{M}}(F)T)) = tr(I_{\mathcal{M}}(E \cap F)T). \tag{5}$$

Now we will describe a quantum system $\mathcal{S}$ by its effect algebra L. To describe a measurement, we need to find a suitable model for the coupling of a quantum system $\mathcal{S}$ described by L with a classical system $\mathcal{A}$ (the measuring apparatus) described by a Boolean σ-algebra B. For this model, we have chosen a bounded Boolean power.

Let L be an effect algebra, and B be a Boolean algebra. Then the tensor product $L \otimes B$ in the category of effect algebras is isomorphic to a bounded Boolean power of L with respect to B. A *bounded Boolean power* of L with respect of B is the set $L \otimes B$ of functions from L to B with finite range and such that

$$f(\ell_1) \wedge f(\ell_2) = 0_B \text{ if } \ell_1 \neq \ell_2 \text{ and } \bigvee_{\ell \in L} f(\ell) = 1_B. \tag{6}$$

We recall that $L \otimes B$ is an effect algebra of the same type as L and that there are full embeddings (i.e., injective morphisms preserving all existing lattice operations) $\lambda : L \to L \otimes B$, where $\lambda(a)(x) = 1_B$ if $x = a$ and $\lambda(a)(x) = 0_B$ if $x \neq a$, and $\beta : B \to L \otimes B$, where $\beta(b)(1_L) = b$, $\beta(b)(0_L) = b'$, and $\beta(b)(x) = 0_B$ if $x \neq 1_L, 0_L$ (Dvurečenskij and Pulmannová, 1994). It turns out that every element f of $L \otimes B$ can be written in the form

$$f = \sum_{i \in K} \lambda(\ell_i) t_i$$

where $(t_i)_{i \in K}$ is a finite partition of unity in B and $(\ell_i))_{i \in K}$ is a finite subset of elements of L, such that

$$f(x) = \bigvee_{i \in K} \lambda(\ell_i)(x) \wedge t_i \tag{7}$$

If all the elements $\ell_i, i \in K$ are different, the above representations are unique.

Let $f = \sum_{k \in K} \lambda(a_k) t_k$, $g = \sum_{j \in J} \lambda(b_j) s_j$ be the unique representations of $f, g \in L \otimes B$. Then $f \perp g$ iff $a_k \perp b_j$ whenever $t_k \wedge s_j \neq_B, k \in K, j \in J$, and $f \oplus g = \sum_{k, j : t_k \wedge s_j \neq 0_B} \lambda(a_k \oplus b_j) t_k \wedge s_j$.

If m is a state on L and μ is a state on B, we can define a (finitely additive) *product state* $m \otimes \mu$ by

$$m \otimes \mu \left(\sum_i \lambda(\ell_i) t_i \right) = \sum m(\ell_i \mu(t_i). \tag{8}$$

Let $(\Omega, \mathcal{F})$ be a measurable space. An $(\Omega, \mathcal{F})$-valued *observable* on L is a σ-morphism from $\mathcal{F}$ to L.

We will assume that a quantum mechanical system $\mathcal{S}$ is described by an (σ-orthocomplete) effect algebra L possessing an ordering set of states $\mathcal{P}_L$.

The measuring apparatus $\mathcal{A}$ is supposed to be a classical object described by a Boolean σ-algebra B with ordering set of states $\mathcal{P}_B$.

The coupled system $\mathcal{S} + \mathcal{A}$ will be described by the bounded Boolean power $L \otimes B$. As a physical state space $\mathcal{P}$ of $\mathcal{S} + \mathcal{A}$ we will consider the convex envelope of the set of all product states, that is, the set of all σ-convex combinations $\sum_i \alpha_1 m_i \otimes \mu_i$, $m_i \in \mathcal{P}_L$, $\mu_i \in \mathcal{P}_B$ and $\alpha_i, i \in \mathbb{N}$, are nonnegative numbers with sum 1.

Assume that we want to measure an observable X on $\mathcal{S}$. Let $(\Omega, \mathcal{F})$ be the value space of X. We choose a measuring apparatus $\mathcal{A}$ described by a Boolean σ-alegebra B and an observable $X_{\mathcal{A}}$ on B with the value space $(\Omega_{\mathcal{A}}, F_{\mathcal{A}})$. If the value spaces of X and $X_{\mathcal{A}}$ are different, we choose a measurable function $f : \Omega \to \Omega_{\mathcal{A}}$ (a pointer function). If the initial state of $\mathcal{S}$ is m and the initial state of $\mathcal{A}$ is $m_{\mathcal{A}}$, the initial state of the coupled system $\mathcal{S} + \mathcal{A}$ will be the product state $m \otimes m_{\mathcal{A}}$. The measurement means an interaction between S and A, which results in a change of the state of the coupled system. This change will be described by a convexity preserving transformation $V : \mathcal{P} \to \mathcal{P}$. If $(m \otimes m_{\mathcal{A}})$ is the final state of $\mathcal{S} + \mathcal{A}$ after the measurement, the restrictions $V(m \otimes m_{\mathcal{A}}) \cdot \lambda$ and $V(m \otimes m_{\mathcal{A}}) \cdot \beta$ uniquely describe the final states of $\mathcal{S}$ and $\mathcal{A}$, respectively.

A 5-tuple $\mathcal{M} = (B, X_{\mathcal{A}}, m_{\mathcal{A}}, f, V)$ will be called a *measurement* of X if the following equality is satisfied:

$$m(X(F)) = V(m \otimes m_{\mathcal{A}} \cdot \beta(X_{\mathcal{A}}(f^{-1}(F))), \forall F \in \mathcal{F}, \forall m \in \mathcal{P}_L. \tag{9}$$

If also the equality

$$m(X(F)) = V(m \otimes m_{\mathcal{A}} \cdot \lambda(X(F)), \forall F \in \mathcal{F}, \forall m \in \mathcal{P}_L. \tag{10}$$

the measurement $\mathcal{M}$ is called of the *first kind*.

For every measurement $\mathcal{M}$, an *instrument* is defined by

$$I_{\mathcal{M}}(F)(m) = V(m \otimes (F)m_{\mathcal{A}})(\beta(X_{\mathcal{A}}(f^{-1}(F))) \wedge \lambda \tag{11}$$

for all $F \in \mathcal{F}$ and all $m \in \mathcal{P}_L$. Here $I_{\mathcal{M}}(F)(m)$ is a σ-additive measure on L, and the state obtained after normalization can be interpreted as the final state on $\mathcal{S}$ after measurement on the condition that the measurement leads to a result in the set F. The instrument reproduces the observable X via the equations

$$I_{\mathcal{M}}(m)(1_L) = m(X(F)), \forall F \in \mathcal{F}, \forall m \in \mathcal{P}_L. \tag{12}$$

A measurement $\mathcal{M}$ is called *repeatable* if for all $E, F \in \mathcal{F}$ and all m,

$$I_{\mathcal{M}}(E)I_{\mathcal{M}}(F)(m)(1_L) = I_{\mathcal{M}}(E \cap F)(m)(1_L). \tag{13}$$

Two measurements $\mathcal{M}_1$ and $\mathcal{M}_2$ are *equivalent* if their instruments are equal.

The basic result is the following.

Theorem 3.1. *For every regular observable X on an effect algebra L there exists a measurement.*

The measurement in the above theorem need not be repeatable. The existence of a repeatable measurement was proved for discrete observables on orthomodular lattices admitting conditional probabilities in (Pulmannová, 1994), where also some other special types of measurements have been studied (see also Tensor Products of Hilbert Space Effect Algebras, Preprint.).

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